S3 Appendix: stability analysis of skyhook and balance control strategies.

This appendix presents the stability analysis for the proposed semi-active controllers. Considering the system dynamics Eq. (13), which takes into account the control input saturation

$$\ddot{x}_p = k\left(x_h - x_p\right) + \left(b_{control} + c\right)\left(\dot{x}_h - \dot{x}_p\right) \tag{S3.1}$$

$b_{control}$ is the damping from the semi-active control command. The semi-active control laws of the skyhook and balance are defined in Eq. (5) and Eq. (3), respectively.

$$b_{sky}(\dot{x}_p, M) = b_1(M) + b_2(M)\frac{\dot{x}_p}{\dot{x}_p - \dot{x}_h} \tag{S3.2}$$

$$b_{bal}(\dot{x}_p, M) = b_d(M) + (k_d(M) - k(M))\frac{x_p - x_h}{\dot{x}_p - \dot{x}_h} \tag{S3.3}$$

The control command $b_{control}$ is obtained using the following saturation due to physical limitations and $b(\dot{x}_p, M)$ from one of the control laws presented in Eq. (2) and Eq. (3).

$$b_{control} = \left\{\begin{matrix} b_{max}, & b(\dot{x}_p, M) \geq b_{max} \\ b_{control}, & b_{max} > b(\dot{x}_p, M) > 0 \\ 0, & 0 \geq b(\dot{x}_p, M) \end{matrix}\right\} \tag{S3.4}$$

The presented stability proof only considers the boundary values of $b_{control}$. Thus both control strategies have the same proof as well as same boundary values.

Let $w = kx_h + (c + b_{control})\dot{x}_h$, then

$$\ddot{x}_p = -\frac{k}{m}x_p - \frac{b_{control} + c}{m}\dot{x}_p + \frac{1}{m}w. \tag{S3.5}$$

The Lyapunov candidate function is defined as

$$V = \mathbf{X^T P X} \tag{S3.6}$$

where X is the state variables and P is a positive matrix

$$\mathbf{X} = \begin{Bmatrix} x_p \\ \dot{x}_p \end{Bmatrix} \quad \mathbf{P} = \begin{bmatrix} \frac{k}{m}(c + b_{max} + m) & 1 \\ 1 & (c + b_{max} + m) \end{bmatrix} \tag{S3.7}$$

For the matrix P be positive definite, the determinant of the matrix P must be greater than zero, $det(P) > 0$. The determinant of P is

$$det(P) = k\left(2b_{max} + 2c + m + \frac{(b_{max}+c)^2}{m}\right) - 1 > 0 \tag{S3.8}$$

The heave compensator has $k >> 1$, $b_{max} >> 1$, $c >> 1$. Thus, the determinant of P is always bigger than 0 and P is a positive definite matrix. Now, the derivative of V is obtained

$$\begin{aligned}\dot{V} = \quad & +2\tfrac{1}{m}w\left(x_p + (c + b_{max} + m)\dot{x}_p\right) - 2\tfrac{k}{m}x_p^2 - 2x_p\dot{x}_p\tfrac{(c+b_{control})}{m} \\ & -2\dot{x}_p^2\left((c + b_{max} + m)\tfrac{(c+b_{control})}{m} - 1\right)\end{aligned} \tag{S3.9}$$

Defining the matrix L

$$\mathbf{L} = \begin{bmatrix} 2\frac{k}{m} & \frac{c+b_{control}}{m} \\ \frac{c+b_{control}}{m} & 2\left[\frac{(c+b_{control})(b_{max}+c+m)}{m} - 1\right] \end{bmatrix} \tag{S3.10}$$

The determinant of L has the below expression, which is always positive, because $c > 1$ and $b_{max} > b_{control} > 0$. Thus L is positive definite.

$$det(\mathbf{L}) = \begin{array}{c} (c + b_{control} - 1)\frac{4k}{m} \\ +\frac{c+b_{control}}{m^2}\left(4k(c + b_{max}) - (c + b_{control})\right) \end{array} > 0 \tag{S3.11}$$

The positive matrix L is used to rewrite the Eq. (**??**)

$$\dot{V} = -\mathbf{X^T L X} + 2\frac{1}{m}w\left(x_p + (c + b_{max} + m)\dot{x}_p\right) \tag{S3.12}$$

When x_h, $\dot{x}_h$ and $b_{control}$ are bounded, w is also bounded, $\|w\| \leq w_0$. We write $x_p + (c + b_{max} + m)\dot{x}_p$ as YX, where $Y = [1, (c + b_{max} + m)]$ and use the inequality $YX \leq \|X\|\,\|Y\|$. The norm of Y is constant. Using the last two inequalities we obtain:

$$\dot{V} \leq -\mathbf{X^T L X} + \frac{2}{m}\|Y\|\,\|X\|\,w_0 \tag{S3.13}$$

The variables λ_{min} and λ_{max} denote the smallest and the largest eigenvalues of a matrix, respectively.

$$\dot{V} \leq -\lambda_{min}(L)\,\|X\|^2 + \frac{2}{m}\|Y\|\,w_0\,\|X\| \tag{S3.14}$$

A part of $-\lambda_{min}(L)\,\|X\|^2$ dominates $\frac{2}{m}\|Y\|\,w_0\,\|X\|$ for large $\|X\|$. The inequality is rewritten as

$$\dot{V} \leq -(1-\theta)\lambda_{min}(L)\left\|X\right\|^2 - \theta\lambda_{min}(L)\left\|X\right\|^2 + \frac{2}{m}\left\|Y\right\| w_0 \left\|X\right\| \tag{S3.15}$$

where $0 < \theta < 1$. Then,

$$\dot{V} \leq -(1-\theta)\lambda_{min}(L)\left\|X\right\|^2, \ \forall \ \left\|X\right\| \geq \frac{2}{m}\left\|Y\right\| w_0 \frac{1}{\theta\lambda_{min}(L)} \tag{S3.16}$$

We conclude that the solutions are globally uniformly ultimately bounded B. The theorem 4.18 of [**?**] is applied to determine the ultimate bound:

$$B = \frac{2}{m}\left\|Y\right\| w_0 \frac{1}{\theta\lambda_{min}(L)}\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \tag{S3.17}$$

References

[1] Khalil H. Nonlinear Systems. 2002.